Section 1.4 The Complex Number System

What is a number?

A number by itself has no meaning. A simplified definition of a number is: *An object that belongs to a group of objects with some shared characteristics.*

As a matter of fact, there are many different kinds of numbers.

Real Numbers:
* Rational
  - -Integers ...-3, $-(1+1)$, $-\frac{5}{5}$, 0, 1, $\frac{4}{2}$, $\sqrt{9}$, ...
  - -NonInteger Rationals ... $-\frac{101}{32}$, ... -1.21, ..., $\frac{1}{4}$, ..., 10.34125
* Irrational

Complex Numbers
* Real Numbers
* Imaginary Numbers

Surreal

Hyperreal numbers

What are the common characteristics shared by all Real numbers?

By defining 1 new number, the number i, which is defined to be the principle square root of -1 we can define a whole new number system. The systems is called the complex numbers.

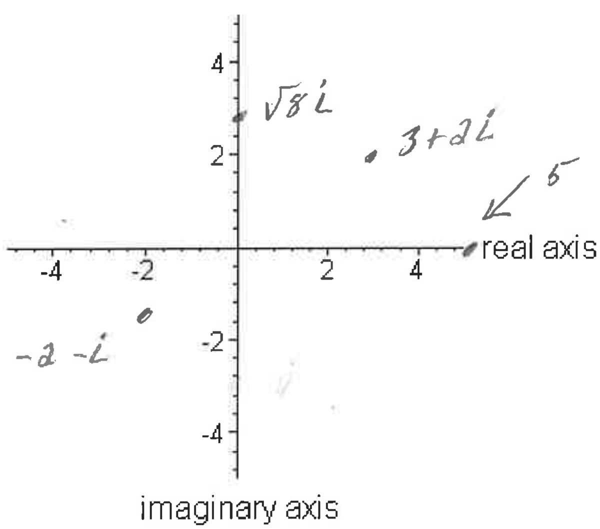
Definition:
The number i is defined such that
$i = \sqrt{-1}$ and $i^2 = -1$.

Definition:
A **complex number** is any number than can be written in the form $a + bi$ where a and b are real numbers.

This definition implies that every complex number has two parts that are defined using real numbers. The same is true of points in a plane. Therefore, every complex number can be paired with a point in a plane. For example:
$1 + 2i$ cooresponds with the point $(1, 2)$
$7 = 7 + 0i$ corresponds with the point $(7, 0)$
$-3i = 0 + (-3i)$ cooresponds with the point $(0, -3)$

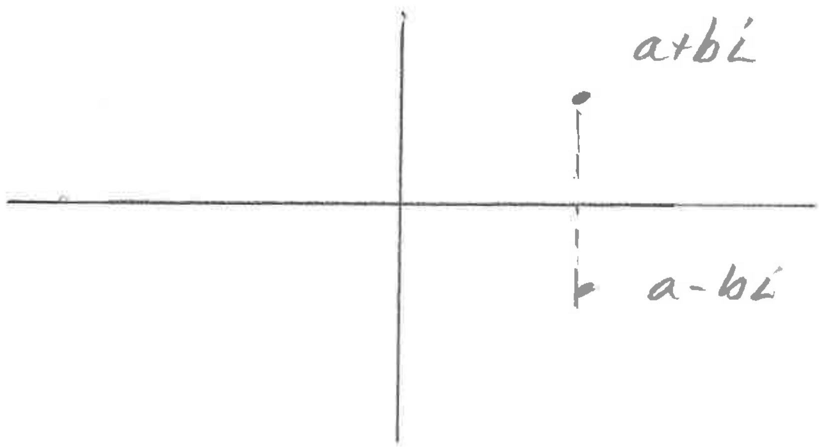
This suggests that whenever we consider an ordered pair of real numbers as ***a number*** then that number is a complex number.

Example:
Plot the numbers below on the complex plane:
$3 + 2i$, $-2 - i$, 5 , $\sqrt{8}\, i$



Definition:
The conjugate of the number $a + bi$ is $a - bi$. The conjugate of $a - bi$ is $a + bi$.

What is the relationship between a complex number and its conjugate if the numbers are viewed as points in a plane?



reflection through the x-axis

What is the sum of a complex number and its conjugate?

$$(a+bi)+(a-bi)$$

$$= 2a$$

What is product of a complex number and its conjugate?

$$(a+bi)(a-bi) = a^2 + abi - abi - b^2 i^2$$

$$= a^2 - b^2 i^2 = a^2 + b^2$$

Complex Number Arithmetic:

To add or subtract complex numbers, simply add the number as you would an expression and combine like terms.

To multiply complex numbers, simply foil the numbers as you would an expression and combine like terms. Remember that $i^2 = -1$.

To divide complex numbers, multiply the top and the bottom of the quotient by the conjugate of the denominator and then reduce.

Example:
Let $\mathbf{u} = 2 + 3i$ and $\mathbf{v} = -1 + i$

Find $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, $\mathbf{uv}$, $\frac{\mathbf{u}}{\mathbf{v}}$ and give each answer in the form $a + bi$.

$$u+v = (2+3i)+(-1+i) = 1+4i$$

$$u-v = (2+3i)-(-1+i) = 3-4i$$

$$uv = (2+3i)(-1+i) = -2-3i+2i-3i^2$$

$$= -2-i-i^2 = -2-i+1 = -1-i$$

$$\frac{u}{v} = \frac{2+3i}{-1+i} \cdot \frac{-1-i}{-1-i} = \frac{-2-2i-3i-3i^2}{1-i^2}$$

$$= \frac{-2-5i+3}{2} = \frac{1-5i}{2} = \frac{1}{2} - \frac{5}{2}i$$

Show that the number $-1 + i$ is a solution to the equation $x^2 + 2x + 2 = 0$.

$$(-1+i)^2 + 2(-1+i) + 2$$

$$= 1 - 2i + i^2 - 2 + 2i + 2$$

$$= 1 - 2i - 1 - 2 + 2i + 2$$

$$= 0$$